Exam Electricity and Magnetism 2

Thursday, November 8, 2007, 14:00-17:00

Before you start, read the following:

- There are 4 problems with a total of 50 points.
- Write your name and student number on every sheet of paper.
- Write the solution of each problem on a separate sheet of paper.
- Illegible writing will be graded as incorrect.
- *Good luck!*

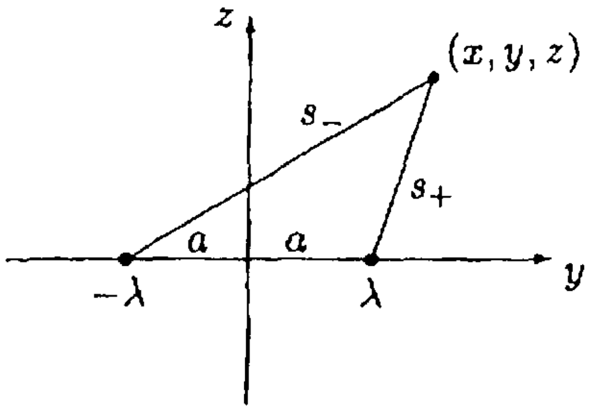
Problem 1 *(40 minutes; 12 points in total)*

An infinitely long straight wire carries a uniform line charge density λ. The distance from the wire is s.

3 pnts (a) Give Gauss's law in integral form and use it to calculate the electric field $\vec{E}$; specify also the direction of $\vec{E}$.

3 pnts (b) Find the potential V as function of s. Explain why you cannot choose the reference point for the potential at $s = \infty$; instead, choose it at $s = a$. Compute the gradient of the potential and check that it yields the correct field.

Next consider two infinitely long wires in the xy-plane, running parallel to the x-axis at $y = \pm a$, that carry uniform line charge densities $+\lambda$ and $-\lambda$, see the figure (the wires are perpendicular to the paper):



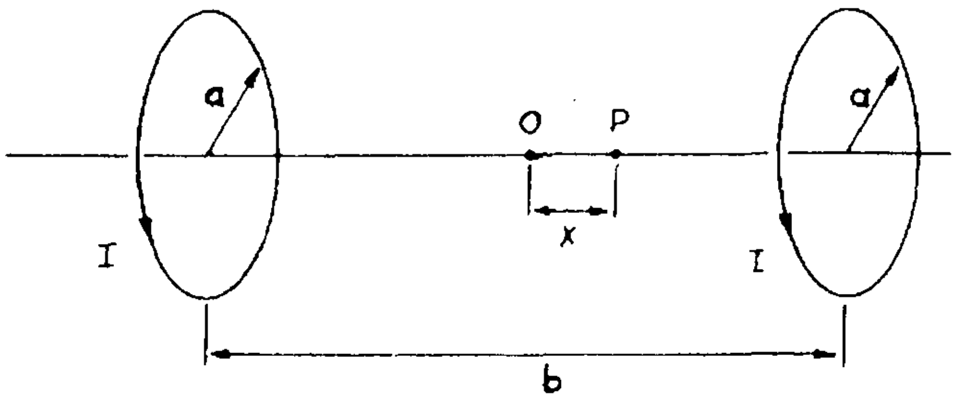
3 pnts (c) Find the potential at any point (x, y, z), using the origin as your reference point. Call $s_\pm$ the distance to $\pm\lambda$; express $s_\pm$ in y and z.

3 pnts (d) Show that the equipotential surfaces are circular cylinders, such that the cross sections with the yz-plane are circles. Locate the axis (y_0, z_0) and radius R of the cylinder that corresponds to a given potential V_0. Define $k = \exp(4\pi\varepsilon_0 V_0/\lambda)$ and express (y_0, z_0) and R in a and k. Sketch the equipotential circles in the yz-plane.

Problem 2 *(40 minutes; 12 points in total)*

3 pnts (a) Give the Biot-Savart law and use it to find the magnetic field at a distance z above the center of a circular loop of radius a which carries a steady current I.

For practical applications, uniform magnetic fields are frequently necessary. One often uses so-called Helmholtz coils: two co-axial loops which carry currents in the same direction. Assume that the coils have their axes on the x-axis, that they have radius a, carry a steady current I each, and are separated by a distance b, see the figure:



3 pnts (b) Find the magnetic field at a point P on the axis of the loops and a distance x from the midpoint O.

3 pnts (c) Expand the expression for the field in a power series retaining terms to order x^2 (use $f(y) = f(0) + yf'(0) + \frac{1}{2}y^2 f''(0) + ...$ for small y).

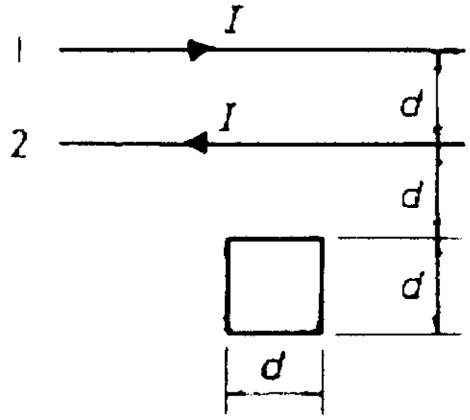
3 pnts (d) What relationship must exist between a and b such that the x^2-terms vanish? What is the significance of this? Show that the field due to the coils to this order and under these conditions is given by

$$B_x = \frac{8\mu_0 I}{5^{3/2} a} .$$

Problem 3 (*40 minutes; 13 points in total*)

3 pnts (a) Give Ampère's law in integral form and use it to calculate the magnetic field due to an infinite wire carrying a current I at a distance s from the wire.

Two infinite parallel wires separated by a distance d carry equal currents I in opposite directions, with I increasing at the rate dI/dt. A square loop of wire of length d on a side lies in the plane of the wires, at a distance d from one of the parallel wires, as illustrated in the figure:



3 pnts (b) Calculate the magnetic flux crossing the square loop due to the magnetic fields from the wires.

4 pnts (c) Give Faraday's law in integral form. Find the emf induced in the square loop. Is the induced current clockwise or anticlockwise? Justify your answer.

3 pnts (d) You have solved the above problem in the so-called quasistatic régime. When is this approximation valid? Discuss *qualitatively* the correct dynamic electromagnetic fields due to the changing current in the wire.

Problem 4 *(40 minutes; 13 points in total)*

A plane wave of (angular) frequency ω travels in the x-direction through vacuum. The electric field is polarized in the y-direction with amplitude E_0:

$$\vec{E}(x,y,z,t) = E_0 \cos(kx - \omega t)\,\hat{y} \ .$$

4 pnts (a) Give Maxwell's equations (in vacuum) in differential form. Show that $\vec{E}(x,y,z,t)$ obeys all four equations and find the associated magnetic field $\vec{B}(x,y,z,t)$. Sketch the $\vec{E}$ and $\vec{B}$ fields in the (x,y,z) coordinate system.

2 pnts (b) Calculate the Poynting vector $\vec{S}$ and average over a full cycle to get the intensity vector $\vec{I}$.

4 pnts (c) The same wave is observed from an inertial system S' moving in the x-direction with speed v relative to the original system. Find the fields $\vec{E}'$ and $\vec{B}'$ in S' and express them in terms of the coordinates (x', y', z', t') in S'. Explain why (or derive) $kx - \omega t = k'x' - \omega' t'$.

3 pnts (d) Find the frequency ω', wavelength λ', and the speed of the waves in S'. What happens to the frequency, amplitude, and intensity of the wave when v approaches c?

Lorentz transformation of the electric field:

$$\begin{aligned} \vec{E}'_{\parallel} &= \vec{E}_{\parallel} \ , \\ \vec{E}'_{\perp} &= \gamma(\vec{E} + \vec{v} \times \vec{B})_{\perp} \ ; \end{aligned}$$

for the magnetic field, replace $\vec{E} \to c\vec{B}$ and $\vec{B} \to -\vec{E}/c$;
$\gamma = (1 - v^2/c^2)^{-1/2}$.

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

Curl :
$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

Divergence : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

Curl : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

Laplacian : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

Second Derivatives

(9) $\nabla\cdot(\nabla\times\mathbf{A}) = 0$

(10) $\nabla\times(\nabla f) = 0$

(11) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}}(\nabla f)\cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int(\nabla\cdot\mathbf{A})\,d\tau = \oint \mathbf{A}\cdot d\mathbf{a}$

Curl Theorem : $\int(\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}$

FUNDAMENTAL CONSTANTS

ϵ_0	=	$8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	=	$4\pi \times 10^{-7}\ \mathrm{N/A^2}$	(permeability of free space)
c	=	$3.00 \times 10^{8}\ \mathrm{m/s}$	(speed of light)
e	=	$1.60 \times 10^{-19}\ \mathrm{C}$	(charge of the electron)
m	=	$9.11 \times 10^{-31}\ \mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\left\{\begin{array}{lcl} x & = & r\sin\theta\cos\phi \\ y & = & r\sin\theta\sin\phi \\ z & = & r\cos\theta \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{x}} & = & \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} & = & \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} & = & \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{array}\right.$$

$$\left\{\begin{array}{lcl} r & = & \sqrt{x^2+y^2+z^2} \\ \theta & = & \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi & = & \tan^{-1}(y/x) \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{r}} & = & \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} & = & \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} & = & -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{array}\right.$$

Cylindrical

$$\left\{\begin{array}{lcl} x & = & s\cos\phi \\ y & = & s\sin\phi \\ z & = & z \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{x}} & = & \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} & = & \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} & = & \hat{\mathbf{z}} \end{array}\right.$$

$$\left\{\begin{array}{lcl} s & = & \sqrt{x^2+y^2} \\ \phi & = & \tan^{-1}(y/x) \\ z & = & z \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{s}} & = & \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} & = & -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} & = & \hat{\mathbf{z}} \end{array}\right.$$